Rendezvous Constants

To meet at a special place, one which is unique amongst all others.

In 1964, O.Gross [1] published a short paper introducing this strange constant. Steven Finch`s book [2] was my first and very informed introduction to this magical constant.

Gross's Theorem concerns closed and bounded subsets of d-dimensional Euclidean space which are connected. The article will concentrate on subsets of $\Re$ and $\Re^2$ where the theorem generates intriguing problems which can be used with students to help them extend their understanding of algebra, geometry and even calculus in more advanced cases. Gross`s theorem is an excellent example of an advanced mathematical result which is in itself surprising. The theorem can be explained to students in the context of new properties of familiar geometrical figures.

Gross's Theorem

For **any** collection of points $x_1, x_2, \ldots\ldots, x_n \in E$, there is a point $y \in E$ for which the average distance from y to $x_1, x_2, \ldots\ldots, x_n$ is $a(E)$, i.e.

$$\frac{1}{n}\sum_{i=1}^{n}\left|x_i - y\right| = a(E) \qquad - \ (1)$$

We emphasize that, although y varies with the collection of points $x_1, x_2, \ldots\ldots, x_n$, the Rendezvous constant, $a(E)$, works for all collections of points and no other constant works.

For many familiar figures, the precise value of $a(E)$ is not known explicitly. When $a(E)$ is known, it's value generally comes from looking at specific arrangements of points for small values of n. In the classroom finding $a(E)$ and finding the value of y for given $x_1, x_2, \ldots\ldots, x_n \in E$ gives scope for students to experience this advanced mathematical result.

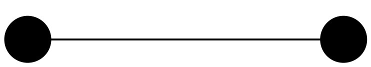

Figure 1a

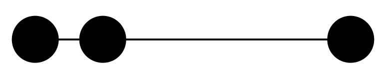

Figure 1b

For example, start with a straight line of length 10 and place a counter on each end so that $n = 2$, $x_1 = 0$ and $x_2 = 10$. See Figure 1a.

Equation (1) then gives: $\frac{1}{2}[(y - 0) + (10 - y)] = a(E)$

Hence $a(E) = 5$

Next use a straight line of length 10, with three counters placed at $x_1 = 0$, $x_2 = 1$, $x_3 = 10$ and $n = 3$. See Figure 1b.

Equation (1) then gives: $\frac{1}{3}[(y-0) + (y-1) + (10-y)] = a(E)$

which with $a(E) = 5$ gives $y = 6$.

Finding the position of y for the Rendezvous Constant on the 1-dimensional number line is an activity suitable for students of all ages and abilities offering challenges, whether played with real or abstract counters. Students can find y for various values of n. While doing this they should make a note of any patterns they find, remembering that the counters need not be placed at distinct points on the number line.

For $n = 2$ we have seen that $a(E) = 5$. You can show the existence of y in E such that $a(E) = 5$ for all n by the following proof:

Let $x_1, x_2, ..., x_n \in E$ and then consider the continuous average distance function

$$f(t) = \frac{1}{n}\sum_{i=1}^{n}|x_i - t|,\ t \in [0,10]$$

As $f(0) = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $f(10) = \frac{1}{n}\sum_{i=1}^{n}|x_i - 10| = 10 - \frac{1}{n}\sum_{i=1}^{n} x_i$

Then either $f(0) \le 5 \le f(10)$ or $f(10) \le 5 \le f(0)$

Applying the Intermediate Value Theorem gives the existence of $y \in [0,10]$ such that $f(y) = 5$. The same argument shows that $a(E) = \frac{1}{2}(p+q)$ for the interval $[p,q]$

The fact that no other constant will work is most surprising! The proof of this uniqueness can be found in [3] using a min-max method, which I will not reproduce here. To give a feel for how this proof is accomplished I will look at a few other cases of the Rendezvous Constant.

Consider the solid 2-dimensional disc with radius $\frac{1}{2}$.

Assume $a(E)$ exists. Firstly let $n = 1$ and $x_1 = 0$ (placed at the centre). Then for any $y \in E$, $a(E) \le \frac{1}{2}$

Next put $n = 2$ and choose diametrically opposite boundary points x_1' and x_2'. Then $|x_2' - x_1'| = 1$, and for any y in E, $\frac{1}{2}[|x_1' - y| + |x_2' - y|] \ge \frac{1}{2}|x_2' - x_1'| = \frac{1}{2}$. So $a(E) \ge \frac{1}{2}$.

Thus $a(E)$ must equal $\frac{1}{2}$ and is unique.

The perimeter of an equilateral triangle with sides of length 1 can be shown to have a Rendezvous Constant of $\dfrac{2+\sqrt{3}}{6}$. This is a 1-dimensional case, as the points are constrained to lie on the perimeter of the triangle.

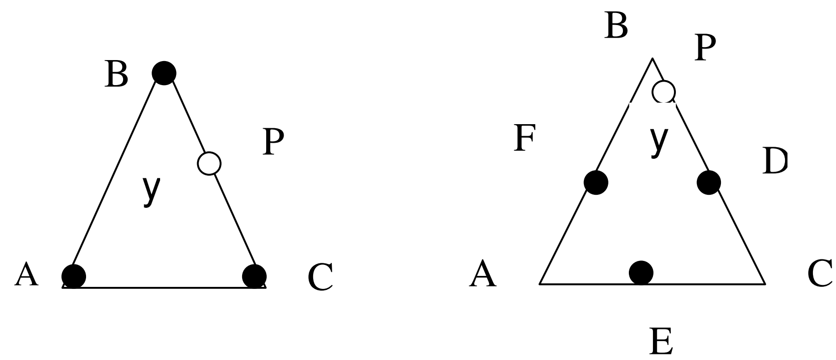

Figure 2a   Figure 2b

Firstly let $n = 3$, with the three points x_1, x_2, x_3 being the vertices A,B,C of the triangle in Figure 2a. Then $\frac{1}{3}\sum_{i=1}^{3}|x_i - y| = \frac{1}{3}(|BP| + |PC| + |AP|) = \frac{1}{3}(1 + |AP|)$. This is a minimum when P is the midpoint of BC, so that $a(E) \geq \frac{2+\sqrt{3}}{6}$

Secondly, let $n = 3$, with the three points x_1, x_2, x_3 being the midpoints D,E,F of each side. See Figure 2b.
Then $\frac{1}{3}\sum_{i=1}^{3}|x_i - y| = \frac{1}{3}(|PE| + |PD| + |PF|) \leq \frac{1}{3}(|PE| + |PD| + |PB| + |BF|) = \frac{1}{3}(1 + |PE|)$.
This is maximum when y is at vertex B, so that $a(E) \leq \frac{2+\sqrt{3}}{6}$. Together this gives the unique Rendezvous Constant as $a(E) = \frac{2+\sqrt{3}}{6}$.

The same method can be used to find the Rendezvous Constant for the perimeter of all regular polygons [3].

Two other interesting 1-dimensional results are:

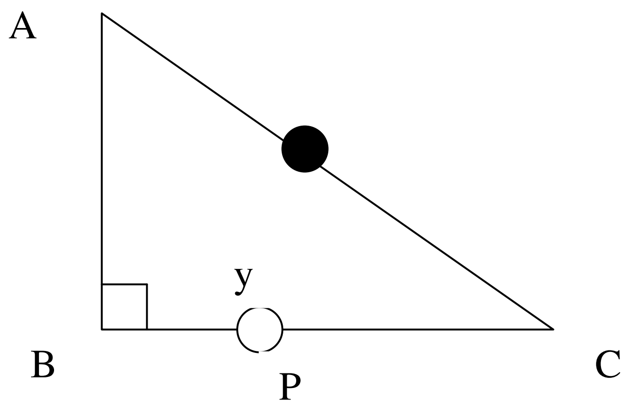

Figure 3

(1) The Rendezvous Constant for the perimeter of a right-angled triangle is equal to half the length of its longest side. First, take $n = 1$ and let x_1 be the midpoint of AC. Then, for any y on the perimeter at P, $|x_1 - y| \le \frac{1}{2}|AC|$,

so $a(E) \le \frac{1}{2}|AC|$. See Figure 3.

Secondly, take n=2 and let x_1, x_2 be at A and C. Then for any y on the perimeter at P, $\frac{1}{2}(|AP| + |PC|) \ge \frac{1}{2}|AC|$, so that $a(E) \ge \frac{1}{2}|AC|$. Together these show that $a(E) = \frac{1}{2}|AC|$.

(2) The circumference of a circle with diameter 1 has $a(E) = \frac{2}{\pi}$. A proof of this can be found by taking the set of points $x_i = (\frac{1}{2}\cos\left(\frac{2\pi i}{n}\right), \frac{1}{2}\sin\left(\frac{2\pi i}{n}\right))$, $i = 0,1,....n-1$ and showing that, for any y in E, $\frac{1}{n}\sum_{i=0}^{n-1}|x_i - y| \to \frac{2}{\pi}$ as $n \to \infty$

Other shapes such as the ellipse evade capture, and no exact formula for $a(E)$ has yet been discovered.

The Rendezvous Constant is an intriguing place for young minds to meet and explore magical maths. I have always found that children of all ages and abilities are fascinated by this result.

<u>References</u>

[1] O.Gross, The Rendezvous Value of Metric Space, **Advances in Game Theory**, ed M.Dresher, L.S. Shapley and A.W. Tucker, Princeton University Press, 1964, pp. 49-53.

[2] Steven R. Finch, **Mathematical Constants**, Cambridge University Press (2003), pp 539-541.

[3] J.Cleary, S.A Morris, and D. Yost, Numerical Geometry – Numbers for Shapes, **Amer. Math. Monthly** 93 (1986) 260-275.

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